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LETTER TO THE EDITOR

Covariance, CPT and the Foldy–Wouthuysen transformation for the Dirac oscillator

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Abstract. The covariance properties and the Foldy-Wouthuysen and Cini-Touschek transformations for the recently proposed Dirac oscillator are found. The charge conjugation, parity and time reversal properties are established. A physical picture as an interaction of an anomalous (chromo) magnetic dipole with a specific (chromo) electric field emerges for this system. This interaction suggests an alternative confinement potential for heavy quarks in QCD.

In a recent work, Moshinsky and Szczepaniak [1] introduced an interesting force in the Dirac equation. In an analogy with the original Dirac procedure, they require an interaction linear in the coordinates. Subsequently, they show that the non-relativistic form of the interaction reduces to one of the harmonic oscillator type. Following [1], we will refer to this system as the Dirac oscillator. Its explicit form is

$$\begin{aligned} \mathrm{i}\partial_t\Psi &= H_{\mathrm{OD}}\Psi \\ &= [\boldsymbol{\alpha}\cdot(\boldsymbol{p}-\mathrm{i}\boldsymbol{r}\beta)m\beta]\Psi \end{aligned} \tag{1}$$

where the usual conventions have been taken [2]. This equation has the remarkable property of having exact solutions [1]†.

In this letter we study equation (1) for the *single-particle case*; we focus on two points. First, we present a manifestly covarient form of equation (1). Second, we deduce for it an exact Foldy-Wouthuysen transformation [4] (FWT). With these two elements a covariant field theory can be defined for the Dirac oscillator. Moreover, from the FWT the energy spectrum and the eigenfunctions are immediately obtained using the non-relativistic harmonic oscillator solutions. In order to build the covariant form of equation (1), we notice that the interaction term in it,

$$\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{r} \tag{2}$$

can be put into the form

$$\sum_{i=1}^{3}\sigma^{0i}x^{i}. \tag{3}$$

Introducing the frame dependent velocity vector

$$u^{\mu}=(1,\boldsymbol{0}) \tag{4}$$

† Exact solutions for a minimal electromagnatic interaction have been known for some time; see [3].

the interaction can be put into the form

$$\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{r}=\sigma^{\mu\nu}x_{\mu}u_{\nu}. \tag{5}$$

Therefore, equation (1) can be rewritten as

$$\left(\not p-m+\kappa\frac{e}{4m}\sigma^{\mu\nu}F_{\mu\nu}\right)\Psi=0 \tag{6}$$

with

$$F^{\mu\nu}=\mu^{\mu}x^{\nu}-\mu^{\nu}x^{\mu} \tag{7}$$

and for the Dirac oscillator $\kappa=2m/e$. This form shows explicitly the covariance of the Dirac oscillator. We remark that the external 'electromagnetic field' of equation (7) is, as usual, frame dependent [5]. Additionally, this form shows a possible physical interpretation of the Dirac oscillator interaction as stemming from the anomalous magnetic moment. It also shows, without further ado, its charge conjugation, parity and time reversal invariance [6], [2, p61].

We can now address to the problem of the FWT. Let us first recall that this transformation defines an operator H_{FW} that is related to the Hamiltonian H by means of a unitary transformation

$$\mathrm{i}\partial_t-H_{\mathrm{FW}}=\mathrm{e}^{\mathrm{i}S}(\mathrm{i}\partial_t-H)\mathrm{e}^{-\mathrm{i}S} \tag{8}$$

where S is to be chosen such that H_{FW} is purely *even*. More explicitly, one splits H into two Hermitian operators: an odd, O_{H}, and an even, E_{H}, part

$$H=O_{\mathrm{H}}+E_{\mathrm{H}}. \tag{9}$$

The usual choice [2] for iS is

$$\mathrm{i}S=BO_{\mathrm{H}}\theta \tag{10}$$

with B and θ Hermitian and even, θ commutes with both B and O_{H}; the additional requirement that $\{B, O_{\mathrm{H}}\}=0$ must be set for consistency so that S is Hermitian. In order to define the FWT one must specify the operator B and the 'angle' θ.

We also remark that, from the well known identity

$$\begin{aligned}A'&=\mathrm{e}^{\mathrm{i}S}A\,\mathrm{e}^{-\mathrm{i}S}\\&=A+[\mathrm{i}S,A]+\tfrac{1}{2!}[\mathrm{i}S,A]]+\ldots\end{aligned} \tag{11}$$

one can show that if S satisfies

$$\{A,\mathrm{i}S\}=0 \tag{12}$$

then

$$A'=A\,\mathrm{e}^{-2\mathrm{i}S} \tag{13}$$

and since for the Dirac oscillator H_{OD} does not depend explicitly on time there are no contributions to the Foldy–Wouthuysen Hamiltonian coming from the transformation of ∂_t. Defining

$$\boldsymbol{\pi}:=\boldsymbol{p}-\mathrm{i}\beta\boldsymbol{r} \tag{14}$$

one can write

$$H_{\mathrm{OD}}=\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+\beta m. \tag{15}$$

If the odd and even parts are defined according to

$$O_{\mathrm{H}} = \boldsymbol{\alpha} \cdot \boldsymbol{\pi} \qquad \text{and} \qquad E_{\mathrm{H}} = \beta m \tag{16}$$

one arrives at the following proposal for the FWT

$$\mathrm{i}S = \beta \boldsymbol{\alpha} \cdot \boldsymbol{\pi}\theta \tag{17}$$

$$= \boldsymbol{\gamma} \cdot \boldsymbol{\pi}\theta. \tag{18}$$

Here $\mathrm{i}S$ anticommutes with O_{H} and E_{H} provided that θ commutes with them. Using the relations

$$\begin{aligned} (\boldsymbol{\gamma} \cdot \boldsymbol{\pi})^2 &= -\boldsymbol{\pi}^\dagger \cdot \boldsymbol{\pi} - \mathrm{i}\boldsymbol{\sigma} \cdot (\boldsymbol{\pi}^\dagger \times \boldsymbol{\pi}) \\ \boldsymbol{\pi}^\dagger \cdot \boldsymbol{\pi} &= \boldsymbol{p}^2 + m^2 - 2\beta \end{aligned} \tag{19}$$

and

$$\boldsymbol{\pi}^\dagger \times \boldsymbol{\pi} = \mathrm{i}2\beta \boldsymbol{L} \tag{20}$$

and defining

$$h^2 := (\boldsymbol{\alpha} \cdot \boldsymbol{\pi})^2 \tag{21}$$

one gets

$$\begin{aligned} h^2 &= -(\boldsymbol{\gamma} \cdot \boldsymbol{\pi})^2 \\ &= \boldsymbol{p}^2 + \boldsymbol{r}^2 - \beta(3 + 4\boldsymbol{L} \cdot \boldsymbol{S}) \end{aligned} \tag{22}$$

which is a positive definite Hermitian operator. The operator h^2 has even and odd roots, two odd roots of h^2 are $\pm(\boldsymbol{\alpha} \cdot \boldsymbol{\pi})$, taking h to be an even root (which we will show explicitly later) we obtain

$$\mathrm{e}^{\mathrm{i}S} = \cos h\theta + (\boldsymbol{\gamma} \cdot \boldsymbol{\pi}) h^{-1} \sin h\theta. \tag{23}$$

From this set-up it follows that

$$\begin{aligned} H_{\mathrm{FW}} &= \mathrm{e}^{\mathrm{i}S} H_{\mathrm{OD}}\, \mathrm{e}^{-\mathrm{i}S} \\ &= \boldsymbol{\alpha} \cdot \boldsymbol{\pi} \left(\cos 2h\theta - \frac{m}{h} \sin 2h\theta \right) + \beta(m \cos 2h\theta + h \sin 2h\theta). \end{aligned} \tag{24}$$

Because the operator h and the mass m are Hermitian it makes sense to choose θ such that

$$\cos 2h\theta - \frac{m}{h} \sin 2h\theta = 0 \tag{25}$$

or, equivalently,

$$\tan 2h\theta = \frac{h}{m}. \tag{26}$$

The last form implies the consistency condition

$$\theta = \theta(h^2). \tag{27}$$

We finally arrive at the desired solution

$$\begin{aligned} H_{\mathrm{FW}} &= \beta\sqrt{m^2+h^2} \\ &= \beta\sqrt{m^2+\boldsymbol{p}^2+\boldsymbol{r}^2-\beta(3+4\boldsymbol{L}\cdot\boldsymbol{S})} \end{aligned} \tag{28}$$

Eigensolutions for this Hamiltonian can easily be obtained because the non-relativistic harmonic oscillator Hamiltonian, β, $\boldsymbol{J}^2$, J_z, $\boldsymbol{L}^2$, $\boldsymbol{S}^2$ and the coordinate reflection operator commute among themselves and with H_{FW}. Furthermore, H_{FW} is a functional of these operators. Hence, the spectra of the Dirac oscillator can be directly obtained from the non-relativistic oscillator solution and the eigenfunctions of these operators [2,3]. These eigensolutions coincide with those first found in [1] by squaring the H_{OD} Hamiltonian for the positive energy part of the spectrum.

Equation (27) also gives an explicit representation of the operator h, which is simply H_{FW} with the mass set equal to zero. Once the negative energy states are defined, one can proceed to construct the appropriate vacuum in order to define a consistent field theory for this system.

One can also use equation (26) to transform the Dirac oscillator to a form adequate for the ultrarelativistic limit. This is done by selecting θ such that the term multiplying β vanishes; the resulting Cini–Touschek [7] Hamiltonian is

$$H_{\mathrm{CT}} = \boldsymbol{\alpha}\cdot\boldsymbol{\pi}\sqrt{1+\frac{m^2}{(\boldsymbol{\alpha}\cdot\boldsymbol{\pi})^2}}. \tag{29}$$

We would like to notice that the remarkably simple properties of the Dirac oscillator interaction and the transparent physical interpretation that follows from its covariant form suggest that it might be a good candidate to be used as the confinement potential in heavy quark systems.

If one tries to find an electric charge distribution that produces the linearly growing electric field that interacts with the anomalous magnetic moment of a Dirac particle to produce a Dirac oscillator, one reaches the peculiar conclusion that it is a uniform (infinite) sphere of charge.

A different conclusion follows in the framework of quantum chromodynamics (QCD) if a slight modification of the usual ideas about confinement is adopted. In QCD, ignoring at this stage the subtleties of Gauss' law for the non-Abelian gauge theories [8], the lines of field are conceivably confined to strings which might simulate the adequate potential without unphysical charge distributions. For example, an effective chromoelectric field which grows with distance stems from a physical picture in which the string is assumed to be of constant volume, as opposed to the more conventional hypothesis of a string of constant cross section.

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